Black Box Analysis of Microwave Networks, Part 2

The K4ERO Loss Formula Untangled

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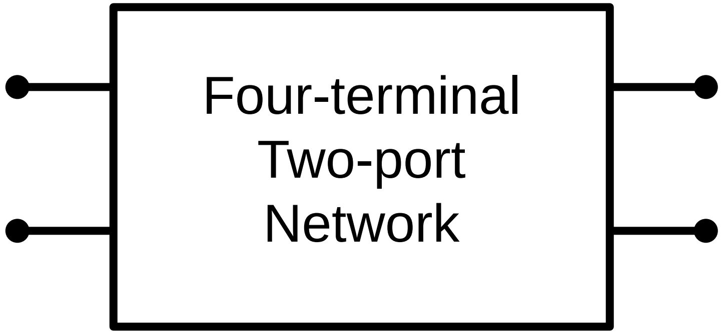
Abstract

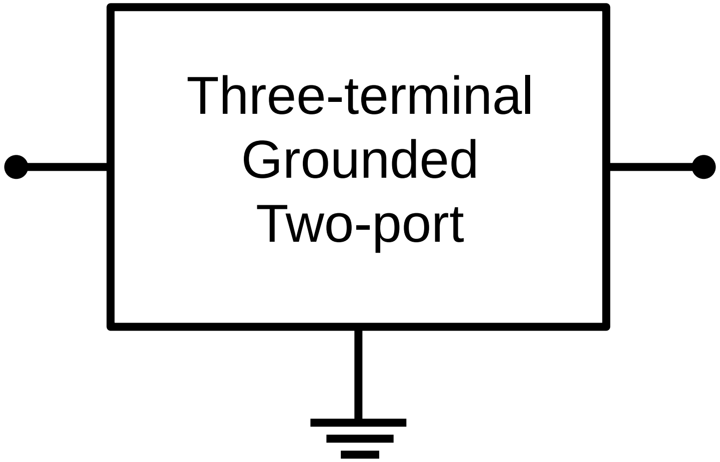
In *QEX*, Jan/Feb 2022, and *QST*, June 2023, John Stanley, K4ERO, proposed a method for determining scattering parameter S_{21} of a 2-port network from VNA measurements made at the network's input port. The method is flawed. It is correct for some cases and wrong for others. Why does Stanley's formula work sometimes and not other times? The mystery is solved here.

Topics

- **Part 1 – Covered previously**
  - Black Boxes
  - External vs Internal behavior
  - Boundary conditions
  - Ports
  - Parameter representations of linear networks
  - *ABCD* parameters
  - Möbius transformation of impedance
  - S parameters
  - Signal flow graphs
  - Conversions between parameters
- **Part 2 – This talk**
  - Brief review of 2-port scattering or “S” parameters
  - K4ERO’s method to determine S_{21}
  - Input reflection coefficient is not S_{11}
  - The key equation
  - Requirements for K4ERO’s method to work
    - Hidden assumptions and restrictions
  - How to correctly determine S_{21} from input port measurements
    - Smith charts
    - Bilinear or Möbius transformations
    - Deschamps method

Two-Ports – Grounded and Ungrounded

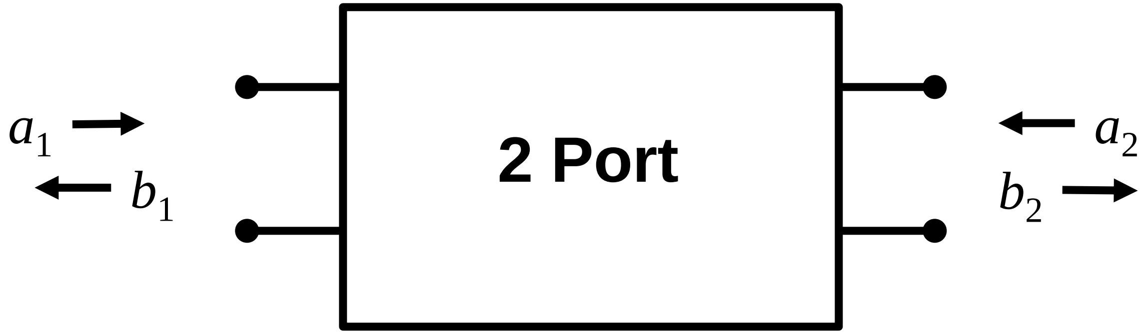




Transmission lines
Filters

Transistors
- Common base
- Common collector
- Common emitter

Black Box Representation in Terms of S Parameters



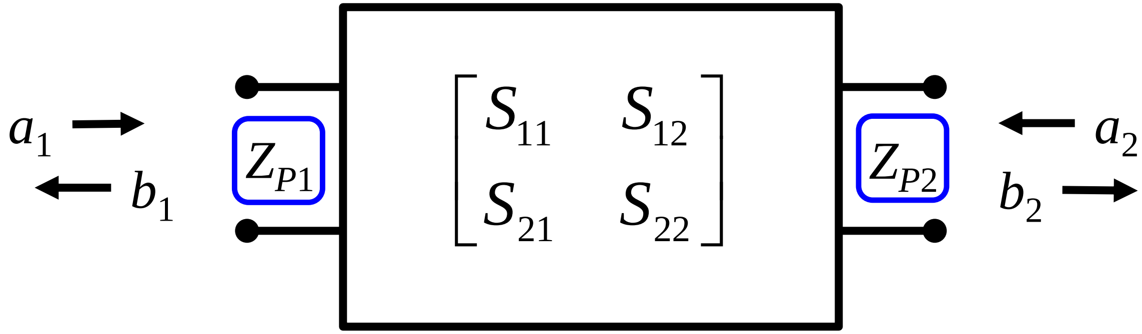
$$b_1 = S_{11}\, a_1 + S_{12}\, a_2$$
$$b_2 = S_{21}\, a_1 + S_{22}\, a_2$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Conservation of energy

Network is lossless ⇔ S is a unitary matrix

Port Reference Impedances



- **Every port has a reference impedance**
- **Reference impedances are arbitrary – not necessarily characteristic impedances or wave impedances**
- **An S parameter specification uses six complex numbers to characterize a 2-port at a single frequency**
  - Two are port reference impedances for Ports 1 and 2
  - Four are S parameters that assume the port reference impedances
- **For multiple frequencies**
  - Port reference impedances are usually specified as constants, independent of frequency
  - S parameters are often tabulated in Touchstone .s2p files

Special Properties

- **Reciprocity**
  - A 2-port is reciprocal if $Z_{P1} = Z_{P2}$ implies $S_{12} = S_{21}$
- **Symmetry**
  - A 2-port is symmetric if $Z_{P1} = Z_{P2}$ implies $S_{12} = S_{21}$ and $S_{11} = S_{22}$
- **Lossless**
  - A 2-port is lossless if and only if **S** is a unitary matrix (complex orthonormal)

$$\mathbf{S}^{-1} = \mathbf{S}^{H}$$ (conjugate transpose or "Hermitian")

- **Lossy**
  - A 2-port is lossy if and only if **S** is not a unitary matrix but rather

$$|S_{11}|^2 + |S_{21}|^2 < 1 \quad \text{and} \quad |S_{22}|^2 + |S_{12}|^2 < 1$$

- **Reflectionless**
  - A 2-port is input reflectionless with matched load at Port 2 if $S_{11} = 0$
  - A 2-port is output reflectionless with matched load at Port 1 if $S_{22} = 0$

A symmetric 2-port can be reversed (turned around) with no effect.

S Parameters Depend on Port Reference Impedances

- If a 2-port is symmetric, there exists a unique value for port impedances Z_{P1} and Z_{P2} that makes S_{11} and S_{22} both zero
- This special impedance is the 2-port's iterative impedance Z_{IT}
- Conversely, if a reciprocal 2-port has $S_{11} = S_{22} = 0$, then changing port impedance Z_{P1} or Z_{P2} will result in $S_{11} \neq 0$ or $S_{22} \neq 0$ or both
- Hence the diagonal elements of S can be made zero or nonzero by choice (or specification) of port reference impedances
- The proof is given later and uses Z-to-S parameter conversion

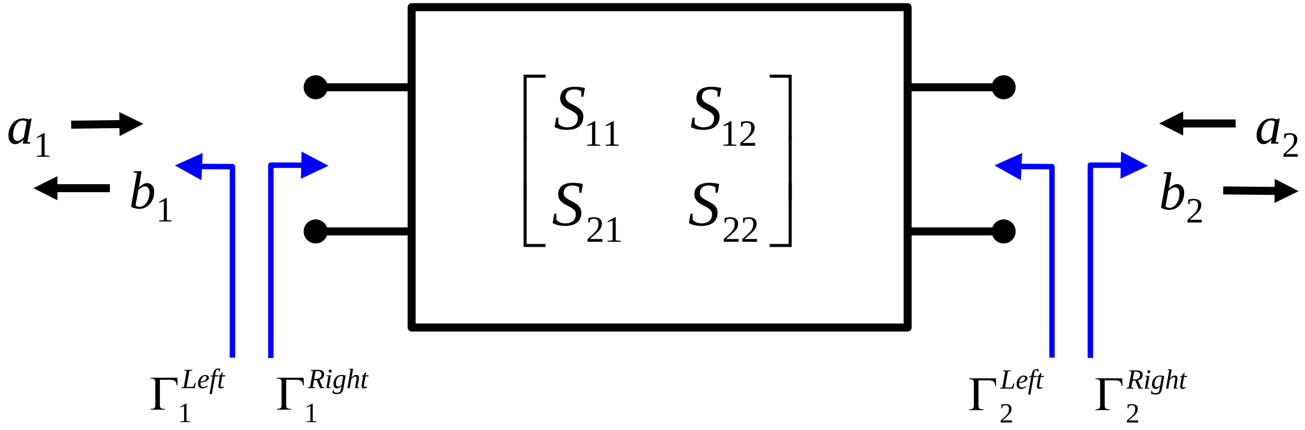
$$\mathbf{S} = \left(\mathbf{Z}_{Port}^{-\frac{1}{2}} \mathbf{Z} \mathbf{Z}_{Port}^{-\frac{1}{2}} + \mathbf{I} \right)^{-1} \left(\mathbf{Z}_{Port}^{-\frac{1}{2}} \mathbf{Z} \mathbf{Z}_{Port}^{-\frac{1}{2}} - \mathbf{I} \right)$$

$$\mathbf{Z} = \mathbf{Z}_{Port}^{\frac{1}{2}} \left(\mathbf{I} - \mathbf{S} \right)^{-1} \left(\mathbf{I} + \mathbf{S} \right) \mathbf{Z}_{Port}^{\frac{1}{2}}$$

where $\mathbf{Z}_{Port}$ is the diagonal matrix of port reference impedances

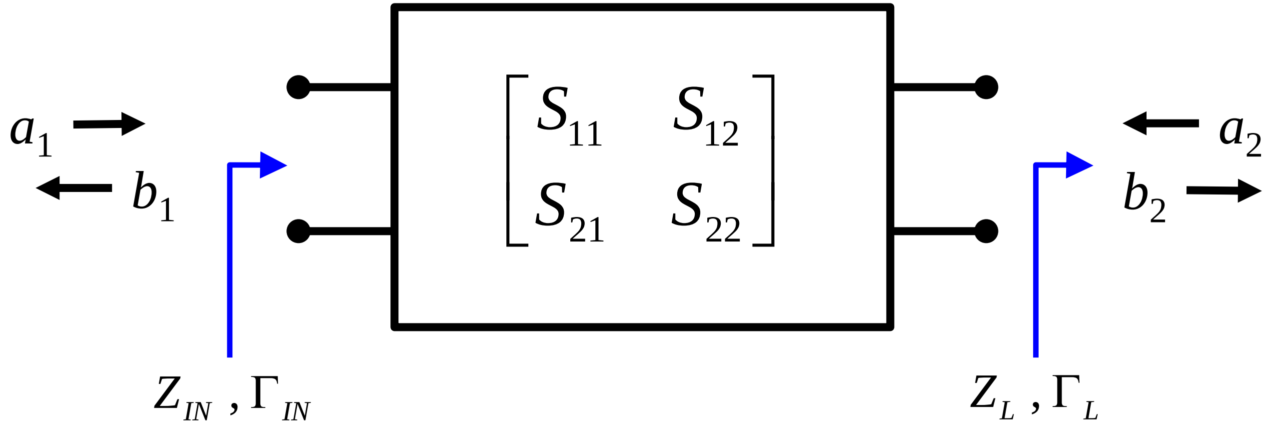
$$\mathbf{Z}_{Port} = \begin{bmatrix} Z_{P1} & 0 \\ 0 & Z_{P2} \end{bmatrix} \quad \text{and} \quad \mathbf{Z}_{Port}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{Z_{P1}} & 0 \\ 0 & \sqrt{Z_{P2}} \end{bmatrix} \quad \text{and} \quad \mathbf{Z}_{Port}^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{\sqrt{Z_{P1}}} & 0 \\ 0 & \frac{1}{\sqrt{Z_{P2}}} \end{bmatrix}$$

Port Reflection Coefficients



- **Reflection coefficients are defined in terms of the port's reference impedance**
- **A port can have both left- and right-facing (or inward and outward) reflection coefficients**
- **The term "reflection" is a misnomer, a carryover from wave mechanics when ports were viewed as waveguide junctions**
- **If a port is connected to a transmission line whose characteristic impedance equals the port's reference impedance, then traveling wave and reflection coefficient interpretations are correct; otherwise not**

Right-Facing Reflection Coefficients (Source on Left)



$$\Gamma_L = \Gamma_2^{Right} = \frac{a_2}{b_2} = \frac{Z_L - Z_{P2}}{Z_L + Z_{P2}}$$

$$\Gamma_{IN} = \Gamma_1^{Right} = \frac{b_1}{a_1} = \frac{Z_{IN} - Z_{P1}}{Z_{IN} + Z_{P1}}$$

Return Loss

$$\text{Return Loss} = \frac{1}{\left|\Gamma_{IN}\right|} = \frac{\left|1 - S_{22}\Gamma_L\right|}{\left|S_{11} - \left(S_{11}S_{22} - S_{12}S_{21}\right)\Gamma_L\right|}$$

$$\text{Return Loss (dB)} = 20\log_{10}\frac{1}{\left|\Gamma_{IN}\right|} = 20\log_{10}\left|\frac{1 - S_{22}\Gamma_L}{S_{11} - \left(S_{11}S_{22} - S_{12}S_{21}\right)\Gamma_L}\right|$$

Insertion Loss and Matched Loss

- **Insertion loss and matched loss are often confused**
- **Insertion loss is defined for 2-ports for specified port reference impedances**
- **Matched loss is defined for transmission lines having attenuation constant "α" nepers per meter (Np/m)**

$$1 \text{ dB/100-ft} = \frac{\ln 10}{20 \times 30.48} \text{ Np/m} = 0.0037772 \text{ Np/m}$$

- **If port reference impedance equals characteristic impedance, insertion loss and matched loss are equal**

$$\text{Insertion Loss} = \frac{1}{|S_{21}|}$$

$$\text{Insertion Loss (dB)} = 20 \log_{10} \frac{1}{|S_{21}|}$$

$$\text{Matched Loss} = e^{\alpha l} = \frac{1}{|S_{21}|} \text{ if } Z_P = Z_0$$

$$\text{Matched Loss (dB)} = 20 \log_{10} e^{\alpha l} = \frac{20}{\ln 10} \ln e^{\alpha l} = 8.6859\, \alpha\, l$$

Reflection Coefficients for Open and Short-Circuit Loads

- **Open-circuit load, $Z_L = \infty$**

$$\Gamma_{L,OC} = \frac{Z_L - Z_{P2}}{Z_L + Z_{P2}} = \frac{\infty - Z_{P2}}{\infty + Z_{P2}} = +1$$

- **Short-circuit load, $Z_L = 0$**

$$\Gamma_{L,SC} = \frac{Z_L - Z_{P2}}{Z_L + Z_{P2}} = \frac{0 - Z_{P2}}{0 + Z_{P2}} = -1$$

- **Input reflection coefficients**

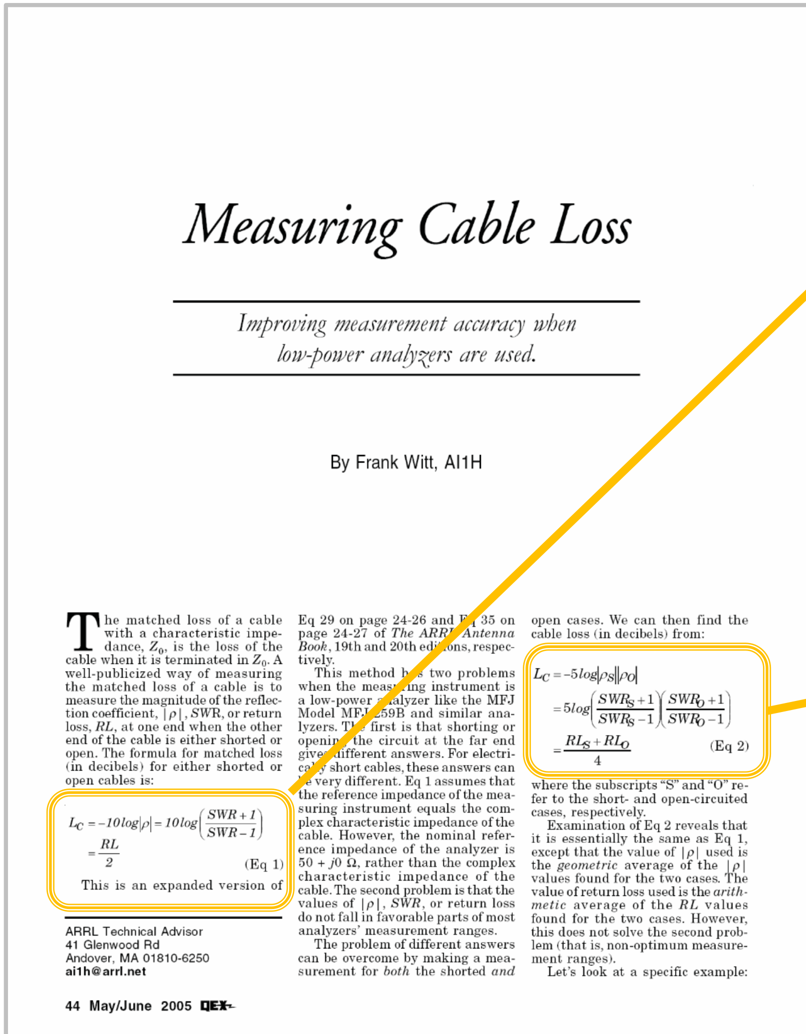
$$\Gamma_{IN,OC} = S_{11} + \frac{S_{12}S_{21}}{1 - S_{22}} \quad \text{and} \quad \Gamma_{IN,SC} = S_{11} - \frac{S_{12}S_{21}}{1 + S_{22}}$$

- **Product**

$$\boxed{\Gamma_{IN,OC}\,\Gamma_{IN,SC} = \left(S_{11} - \frac{S_{12}S_{21}}{1 + S_{22}} \right)\left(S_{11} + \frac{S_{12}S_{21}}{1 - S_{22}} \right)}$$

Frank Witt, AI1H

QEX, May/June 2005



Measuring Cable Loss

Improving measurement accuracy when low-power analyzers are used.

By Frank Witt, AI1H

The matched loss of a cable with a characteristic impedance, Z_0, is the loss of the cable when it is terminated in Z_0. A well-publicized way of measuring the matched loss of a cable is to measure the magnitude of the reflection coefficient, $|\rho|$, *SWR*, or return loss, *RL*, at one end when the other end of the cable is either shorted or open. The formula for matched loss (in decibels) for either shorted or open cables is:

$$L_C = -10\log|\rho| = 10\log\left(\frac{SWR+1}{SWR-1}\right) = \frac{RL}{2} \quad \text{(Eq 1)}$$

This is an expanded version of Eq 29 on page 24-26 and Eq 35 on page 24-27 of *The ARRL Antenna Book*, 19th and 20th editions, respectively.

This method has two problems when the measuring instrument is a low-power analyzer like the MFJ Model MFJ-259B and similar analyzers. The first is that shorting or opening the circuit at the far end gives different answers. For electrically short cables, these answers can be very different. Eq 1 assumes that the reference impedance of the measuring instrument equals the complex characteristic impedance of the cable. However, the nominal reference impedance of the analyzer is $50 + j0\ \Omega$, rather than the complex characteristic impedance of the cable. The second problem is that the values of $|\rho|$, *SWR*, or return loss do not fall in favorable parts of most analyzers' measurement ranges.

The problem of different answers can be overcome by making a measurement for *both* the shorted *and* open cases. We can then find the cable loss (in decibels) from:

$$L_C = -5\log|\rho_S||\rho_O| = 5\log\left(\frac{SWR_S+1}{SWR_S-1}\right)\left(\frac{SWR_O+1}{SWR_O-1}\right) = \frac{RL_S+RL_O}{4} \quad \text{(Eq 2)}$$

where the subscripts "S" and "O" refer to the short- and open-circuited cases, respectively.

Examination of Eq 2 reveals that it is essentially the same as Eq 1, except that the value of $|\rho|$ used is the *geometric* average of the $|\rho|$ values found for the two cases. The value of return loss used is the *arithmetic* average of the *RL* values found for the two cases. However, this does not solve the second problem (that is, non-optimum measurement ranges).

Let's look at a specific example:

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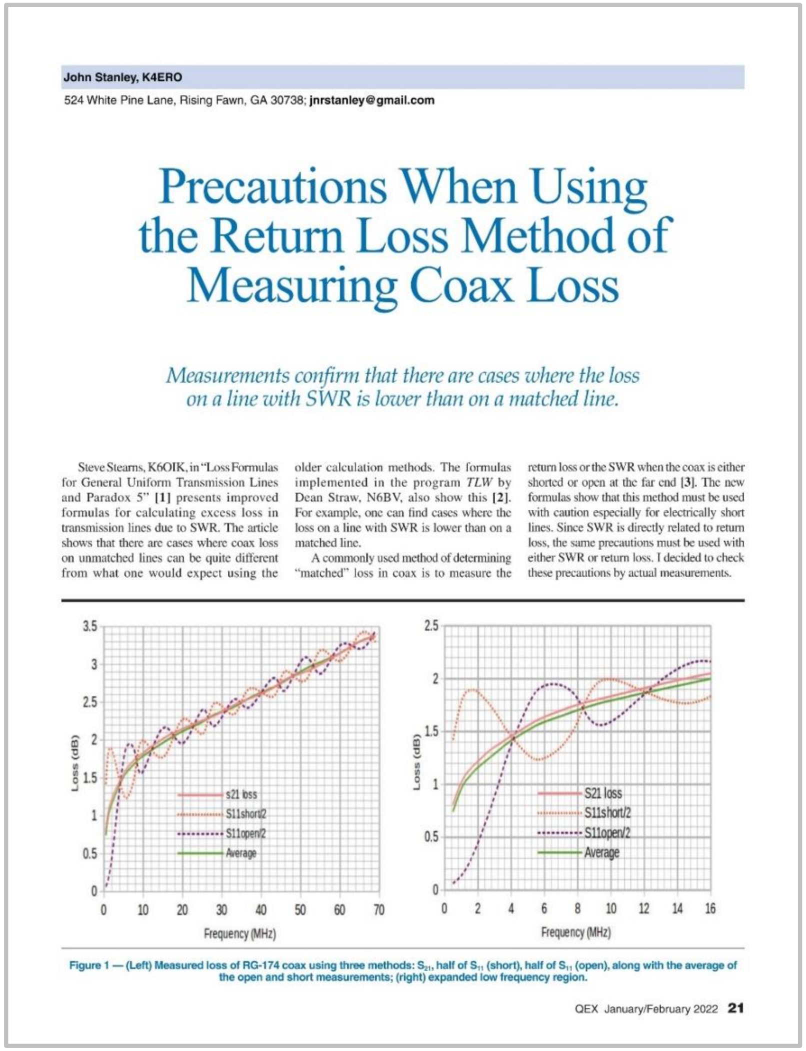
$$L_C = -10\log|\rho| = 10\log\left(\frac{SWR+1}{SWR-1}\right) = \frac{RL}{2} = \text{Matched Loss} = \frac{1}{|S_{21}|}$$

$$L_C = -5\log|\rho_S||\rho_O| = 5\log\left(\frac{SWR_S+1}{SWR_S-1}\right)\left(\frac{SWR_O+1}{SWR_O-1}\right) = \frac{RL_S+RL_O}{4}$$

Two Articles Motivated this Presentation

QEX, January/February 2022



John Stanley, K4ERO

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Precautions When Using the Return Loss Method of Measuring Coax Loss

Measurements confirm that there are cases where the loss on a line with SWR is lower than on a matched line.

Steve Stearns, K6OIK, in "Loss Formulas for General Uniform Transmission Lines and Paradox 5" **[1]** presents improved formulas for calculating excess loss in transmission lines due to SWR. The article shows that there are cases where coax loss on unmatched lines can be quite different from what one would expect using the older calculation methods. The formulas implemented in the program *TLW* by Dean Straw, N6BV, also show this **[2]**. For example, one can find cases where the loss on a line with SWR is lower than on a matched line.

A commonly used method of determining "matched" loss in coax is to measure the return loss or the SWR when the coax is either shorted or open at the far end **[3]**. The new formulas show that this method must be used with caution especially for electrically short lines. Since SWR is directly related to return loss, the same precautions must be used with either SWR or return loss. I decided to check these precautions by actual measurements.

Figure 1 — (Left) Measured loss of RG-174 coax using three methods: S_{21}, half of S_{11} (short), half of S_{11} (open), along with the average of the open and short measurements; (right) expanded low frequency region.

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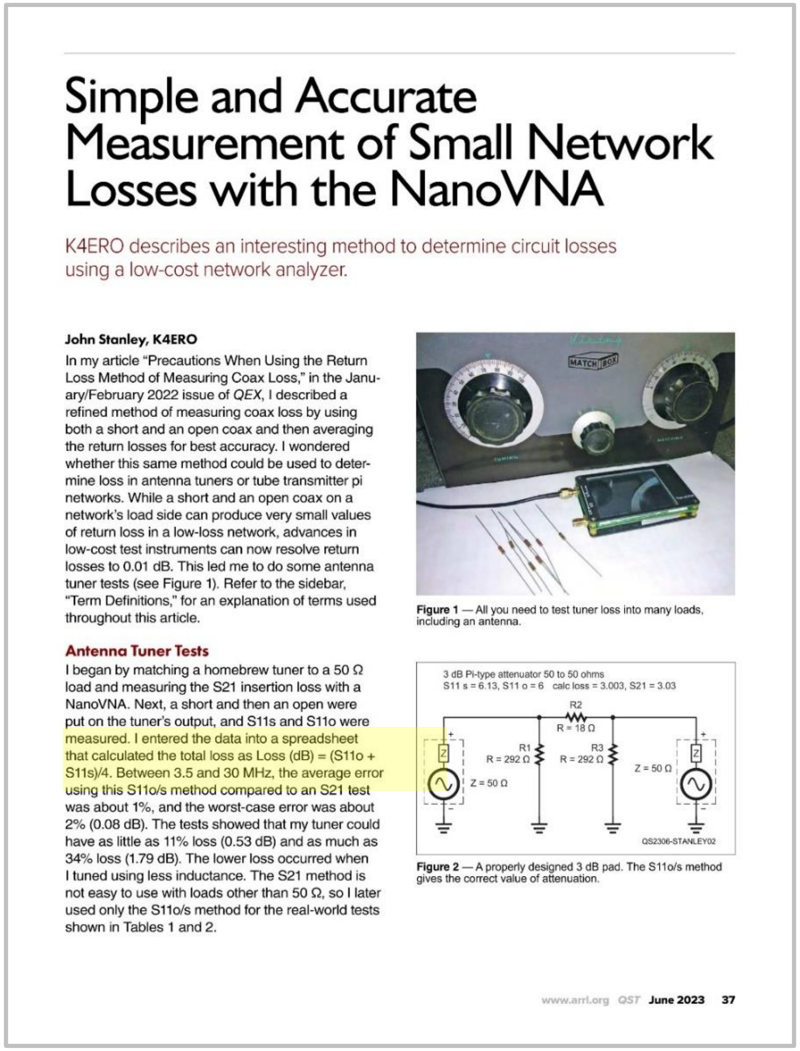
QST, June 2023



Simple and Accurate Measurement of Small Network Losses with the NanoVNA

K4ERO describes an interesting method to determine circuit losses using a low-cost network analyzer.

John Stanley, K4ERO

In my article "Precautions When Using the Return Loss Method of Measuring Coax Loss," in the January/February 2022 issue of *QEX*, I described a refined method of measuring coax loss by using both a short and an open coax and then averaging the return losses for best accuracy. I wondered whether this same method could be used to determine loss in antenna tuners or tube transmitter pi networks. While a short and an open coax on a network's load side can produce very small values of return loss in a low-loss network, advances in low-cost test instruments can now resolve return losses to 0.01 dB. This led me to do some antenna tuner tests (see Figure 1). Refer to the sidebar, "Term Definitions," for an explanation of terms used throughout this article.

Antenna Tuner Tests

I began by matching a homebrew tuner to a 50 Ω load and measuring the S21 insertion loss with a NanoVNA. Next, a short and then an open were put on the tuner's output, and S11s and S11o were measured. I entered the data into a spreadsheet that calculated the total loss as Loss (dB) = (S11o + S11s)/4. Between 3.5 and 30 MHz, the average error using this S11o/s method compared to an S21 test was about 1%, and the worst-case error was about 2% (0.08 dB). The tests showed that my tuner could have as little as 11% loss (0.53 dB) and as much as 34% loss (1.79 dB). The lower loss occurred when I tuned using less inductance. The S21 method is not easy to use with loads other than 50 Ω, so I later used only the S11o/s method for the real-world tests shown in Tables 1 and 2.

Figure 1 — All you need to test tuner loss into many loads, including an antenna.

Figure 2 — A properly designed 3 dB pad. The S11o/s method gives the correct value of attenuation.

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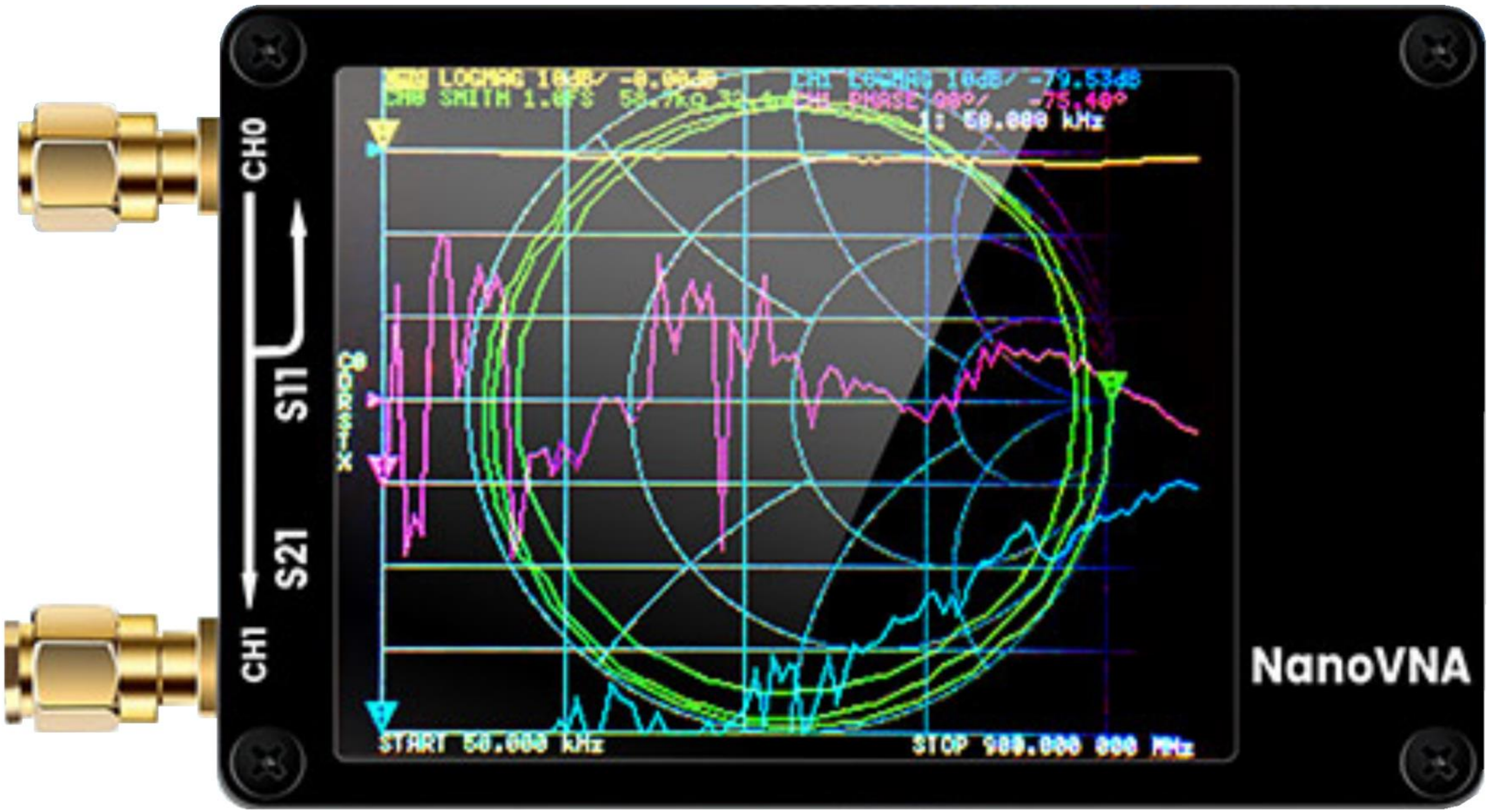
What Can You Measure?

NanoVNA-H4, circa 2020



A popular raffle prize at FARS member meetings!

A Professional Unit

Keysight/Agilent N9913 FieldFox, circa 2012

Unfortunately, not a raffle prize at FARS member meetings!

The Claims

"… If we average the short and open measured data we get a value very close to the value measured by the S_{21} method. The averaged values as shown on the graph fall on the line…"

— John Stanley, K4ERO, *QEX*, Jan/Feb 2022

"… Next, a short and then an open were put on the tuner's output, and S11s and S11o were measured. I entered the data into a spreadsheet that calculated the total loss as Loss (dB) = (S11o + S11s)/4. Between 3.5 and 20 MHz, the average error using the S11o/s method compared to an S21 test was about 1%, and the worst-case error was about 2% (0.08 dB)…"

— John Stanley, K4ERO, *QST*, June 2023

$$\text{Loss (dB)} = \frac{S11o(\text{dB}) + S11s(\text{dB})}{2}$$

$$\text{Total Loss (dB)} = \frac{S11o(\text{dB}) + S11s(\text{dB})}{4}$$

$$\text{Loss (dB)} = \frac{-20\log_{10}\left(S11o\right) - 20\log_{10}\left(S11s\right)}{2}$$

$$\text{Total Loss (dB)} = \frac{-20\log_{10}\left(S11o\right) - 20\log_{10}\left(S11s\right)}{4}$$

So which formula, if any, is right?
And why?

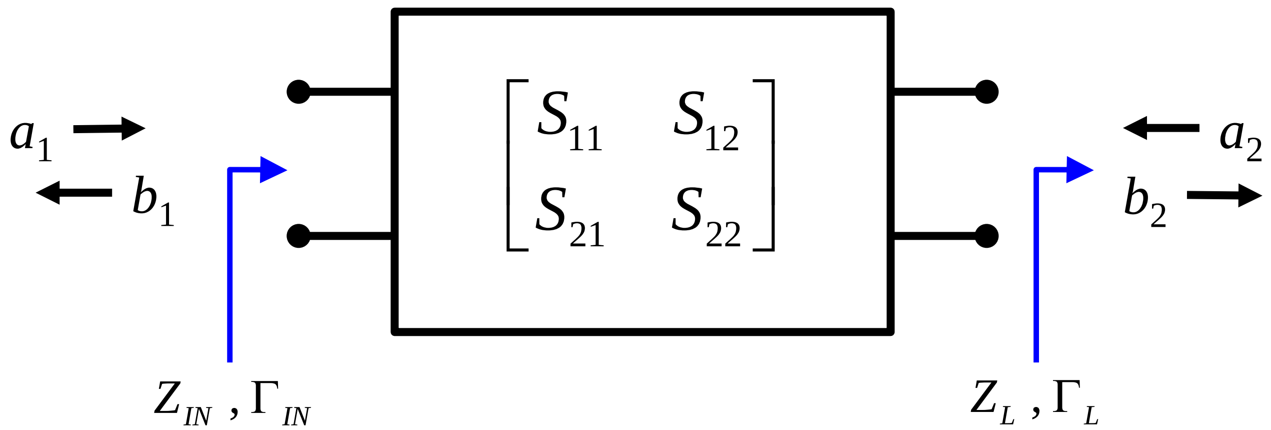
Confusions

- **The term "loss" is not qualified or given a precise definition**
  - Return loss, insertion loss, matched loss, total loss
- **The symbols "S11o" and "S11s" are not scattering parameters of any network**
- **They are simply the magnitudes of the input reflection coefficients measured relative to the VNA reference impedance of 50 ohms**

$$S11o \triangleq \left|\Gamma_{IN,O}\right| \qquad \text{and} \qquad S11s \triangleq \left|\Gamma_{IN,S}\right|$$

- **K4ERO claims**
  - Linear units $$\left|S_{21}\right| = \sqrt[4]{S11o\ S11s}$$
  - dB units $$\left|S_{21}\right|(\text{dB}) = \frac{S11o(\text{dB}) + S11s(\text{dB})}{4}$$

The Key Equation



$$\Gamma_{IN} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$$

$$= \frac{S_{11} - \left(S_{11}S_{22} - S_{12}S_{21}\right)\Gamma_L}{1 - S_{22}\Gamma_L}$$

Bilinear or Möbius transformation

Proof is by signal flow graph theory and Mason's rule

D.M. Pozar, *Microwave Engineering*, 4e, Wiley, 2012, p. 197.

R.E. Collin, *Foundations for Microwave Engineering*, 2e, Wiley, 2001, p. 254.

S. Ramo, et al., *Fields and Waves in Communication Electronics*, 3e, Wiley, 1994, p. 543.

Derivation of K4ERO's Generalization of AI1H

Key equation

$$\Gamma_{IN} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$$

$$\Gamma_{IN,OC} = S_{11} + \frac{S_{12}S_{21}}{1 - S_{22}}$$

$$\Gamma_{IN,SC} = S_{11} - \frac{S_{12}S_{21}}{1 + S_{22}}$$

Assumptions

$$S_{12} = S_{21} \quad \text{and} \quad S_{11} = S_{22} = 0$$

$$\Gamma_{IN,OC} = S_{21}^2$$

$$\Gamma_{IN,SC} = -S_{21}^2$$

$$\left|\Gamma_{IN,OC}\right|\left|\Gamma_{IN,SC}\right| = \left|S_{21}\right|^4$$

Result

$$\left|S_{21}\right| = \sqrt[4]{\left|\Gamma_{IN,OC}\right|\left|\Gamma_{IN,SC}\right|}$$

Example 1: 50-ohm Transmission Line

- **The 50-ohm S matrix and input reflection coefficients are**

$$\mathbf{S}_{50} = \begin{bmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{bmatrix}$$

$$\Gamma_{IN,OC} = e^{-2\gamma l} \quad \text{and} \quad \Gamma_{IN,SC} = -e^{-2\gamma l}$$

- **The K4ERO loss formula agrees with Witt, AI1H, and insertion loss equals matched loss**

$$\begin{aligned} |S_{21}| &= \sqrt[4]{|\Gamma_{IN,OC}|\,|\Gamma_{IN,SC}|} \\ &= \sqrt[4]{|e^{-4\gamma l}|} \\ &= e^{-\alpha l} \end{aligned}$$

Example 2: 450-ohm Transmission Line

- **Consider a 50-ohm VNA measuring a 450-ohm transmission line segment**

$$\mathbf{S}_{450} = \begin{bmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{bmatrix}$$

- **Converting S from 450-ohm to 50-ohm reference gives**

$$\mathbf{S}_{50} = \begin{bmatrix} \dfrac{40 \sinh \gamma l}{8 \cosh \gamma l + 41 \sinh \gamma l} & \dfrac{9}{8 \cosh \gamma l + 41 \sinh \gamma l} \\ \dfrac{9}{8 \cosh \gamma l + 41 \sinh \gamma l} & \dfrac{40 \sinh \gamma l}{8 \cosh \gamma l + 41 \sinh \gamma l} \end{bmatrix}$$

- **Note that S_{11} and S_{22} now depend on length**

P.A. Rizzi, *Microwave Engineering Passive Circuits*, Prentice Hall, 1988, p. 547.

450-ohm Transmission Line (Continued)

- **Assume lossless line, $\alpha = 0$**

$$\cosh(\gamma l) = \cos(\beta l)$$
$$\sinh(\gamma l) = j \sin(\beta l)$$

- **S matrix for lossless 450-ohm line referenced to 50 ohms**

$$\mathbf{S}_{50} = \begin{bmatrix} \dfrac{j40 \sin \beta l}{8 \cos \beta l + j41 \sin \beta l} & \dfrac{9}{8 \cos \beta l + j41 \sin \beta l} \\ \dfrac{9}{8 \cos \beta l + j41 \sin \beta l} & \dfrac{j40 \sin \beta l}{8 \cos \beta l + j41 \sin \beta l} \end{bmatrix}$$

- **Assume line length is quarter-wave, $\beta l = \pi/2$**

$$\mathbf{S}_{50} = \begin{bmatrix} \dfrac{40}{41} & -j\dfrac{9}{41} \\ -j\dfrac{9}{41} & \dfrac{40}{41} \end{bmatrix}$$

450-ohm Transmission Line (Continued)

- **The input reflection coefficients for open and shorted loads are**

$$\Gamma_{IN,OC} = S_{11} + \frac{S_{12}S_{21}}{1-S_{22}} = \frac{40}{41} + \frac{-81}{41\times 39} = \frac{1479}{1599}$$

$$\Gamma_{IN,SC} = S_{11} - \frac{S_{12}S_{21}}{1+S_{22}} = \frac{40}{41} - \frac{-1}{41} = 1$$

- **The K4ERO loss formula predicts**

$$|S_{21}| = \sqrt[4]{|\Gamma_{IN,OC}||\Gamma_{IN,SC}|} = \sqrt[4]{\frac{1479}{1599}} = 0.98069 \neq \frac{9}{41} = 0.21951$$

- **Error is 13 dB!**

The K4ERO method fails for transmission line sections whose characteristic impedance is different from 50-ohm VNA impedance.

Example 3: A Reciprocal Non-Symmetric 2-Port

- Consider a reciprocal 2-port whose scattering matrix does NOT satisfy

$$S_{11} = S_{22} = 0$$

- Let S be the unitary matrix

$$\mathbf{S} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

- The open and shorted input reflection coefficients are

$$\Gamma_{IN,OC} = \sqrt{2} - 1 \quad \text{and} \quad \Gamma_{IN,SC} = \sqrt{2} + 1$$

- The K4ERO loss formula predicts

$$\left|S_{21}\right| = \sqrt[4]{1} = 1 \neq \frac{1}{\sqrt{2}} = 0.7071$$

Error is 3 dB

Example 4: A Reciprocal, Symmetric 2-Port

- Consider a reciprocal, symmetric 2-port whose scattering matrix does NOT satisfy

$$S_{11} = S_{22} = 0$$

- Let S be the unitary matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The open and shorted input reflection coefficients are

$$\Gamma_{IN,OC} = 1 \quad \text{and} \quad \Gamma_{IN,SC} = 1$$

- The K4ERO loss formula predicts

$$\left|S_{21}\right| = \sqrt[4]{1} = 1 \neq 0$$

Error is infinite dB!

Example 5: A Reciprocal Non-Symmetric 2-Port

- **Consider a reciprocal, non-symmetric 2-port whose scattering matrix does NOT satisfy**

$$S_{11} = S_{22} = 0$$

- **Let S be the unitary matrix**

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}$$

- **The open and shorted input reflection coefficients are**

$$\Gamma_{IN,OC} = 1 \quad \text{and} \quad \Gamma_{IN,SC} = 1$$

- **The K4ERO loss formula predicts**

$$\left|S_{21}\right| = \sqrt[4]{1} = 1 \neq 0$$

Error is infinite dB!

Comments on the Validity of the K4ERO Method

- **The loss formula is correct if certain conditions are met**
- **The port reference and VNA impedances are equal and the following conditions hold at this impedance**
  - Reciprocity: $S_{21} = S_{12}$
  - Symmetry: $S_{11} = S_{22}$
  - $S_{11} = 0$ and $S_{22} = 0$

- **The K4ERO method fails if some but not all conditions are met**
- **If a 2-port satisfies $S_{11} = S_{22} = 0$ but is not reciprocal, the method gives the geometric mean of $|S_{12}|$ and $|S_{21}|$ instead of $|S_{21}|$**

$$\sqrt{\left|S_{12}\right|\left|S_{21}\right|} = \sqrt[4]{\left|\Gamma_{IN,OC}\right|\left|\Gamma_{IN,SC}\right|}$$

An Interesting Fact

- **Theorem: There exists a unique port impedance Z_P for which the S matrix has zero diagonal**
- **Proof: Consider Z-to-S parameter conversion**

$$S_{11} = \frac{\left(Z_{11} - Z_{P1}\right)\left(Z_{22} + Z_{P2}\right) - Z_{12}Z_{21}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \qquad S_{12} = \frac{2Z_{12}\sqrt{Z_{P1}Z_{P2}}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$

$$S_{21} = \frac{2Z_{21}\sqrt{Z_{P1}Z_{P2}}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \qquad S_{22} = \frac{\left(Z_{11} + Z_{P1}\right)\left(Z_{22} - Z_{P2}\right) - Z_{12}Z_{21}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$

- **Set the diagonal elements to zero**

$$S_{11} = 0 \qquad \Leftrightarrow \qquad \left(Z_{11} - Z_{P1}\right)\left(Z_{22} + Z_{P1}\right) - Z_{12}Z_{21} = 0$$

$$S_{22} = 0 \qquad \Leftrightarrow \qquad \left(Z_{11} + Z_{P2}\right)\left(Z_{22} - Z_{P2}\right) - Z_{12}Z_{21} = 0$$

Interesting Fact (Continiued)

- **Expand both equations to get two quadratic equations**

$$\begin{aligned} 0 &= \left(Z_{11} - Z_{P1}\right)\left(Z_{22} + Z_{P1}\right) - Z_{12}Z_{21} \\ &= Z_{P1}^2 - \left(Z_{11} - Z_{22}\right)Z_{P1} - \left(Z_{11}Z_{22} - Z_{12}Z_{21}\right) \end{aligned}$$

and

$$\begin{aligned} 0 &= \left(Z_{11} + Z_{P2}\right)\left(Z_{22} - Z_{P2}\right) - Z_{12}Z_{21} \\ &= Z_{P2}^2 + \left(Z_{11} - Z_{22}\right)Z_{P2} - \left(Z_{11}Z_{22} - Z_{12}Z_{21}\right) \end{aligned}$$

- **Since $S_{11} = S_{22}$, the network is symmetric, $Z_{11} = Z_{22}$, and the middle terms drop out**
- **Both quadratic equations then have the same solution Z_P**

$$Z_P = \pm\sqrt{Z_{11}Z_{22} - Z_{12}Z_{21}} = \pm\sqrt{\det \mathbf{Z}} = \pm\sqrt{\frac{B}{C}}$$

Iterative impedance

- **The "+" sign is chosen to get positive real part**
- **Conclusion 1: There is always a port impedance for which the S matrix has zero diagonal**
- **Conclusion 2: This special impedance is the 2-port's iterative impedance Z_{IT}, which was discussed in Part 1**

A Comment on the Validity of the K4ERO Method

- **When using a 50-ohm VNA, K4ERO's method and formula are correct provided it is known *a priori* that**
  - The network under test is reciprocal and symmetric
  - The network's 50-ohm scattering matrix has zero diagonal

$$S_{11} = S_{22} = 0$$

  - Or equivalently, the network's iterative impedance is equal to the VNA impedance of 50-ohms

$$Z_{IT} = \sqrt{\det \mathbf{Z}} = \sqrt{Z_{11}Z_{22} - Z_{21}^2} = 50 \text{ ohms}$$

How to Determine All S Parameters from Input Port Measurements

A venerable 71-year old method

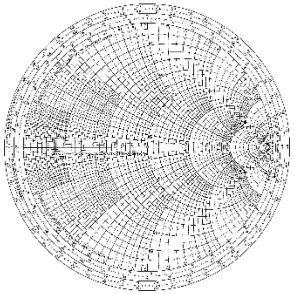
Tools Needed – Post WWII, Pre-Computer Era

- RF impedance bridge (or 1-port VNA)

Heathkit QM-1
circa 1953

General Radio 1606
R-F Bridge
A: circa 1955
B: circa 1967

- Smith chart



Smith chart
circa 1939

- Compass

- Straight edge

Euclid
circa 300 B.C.

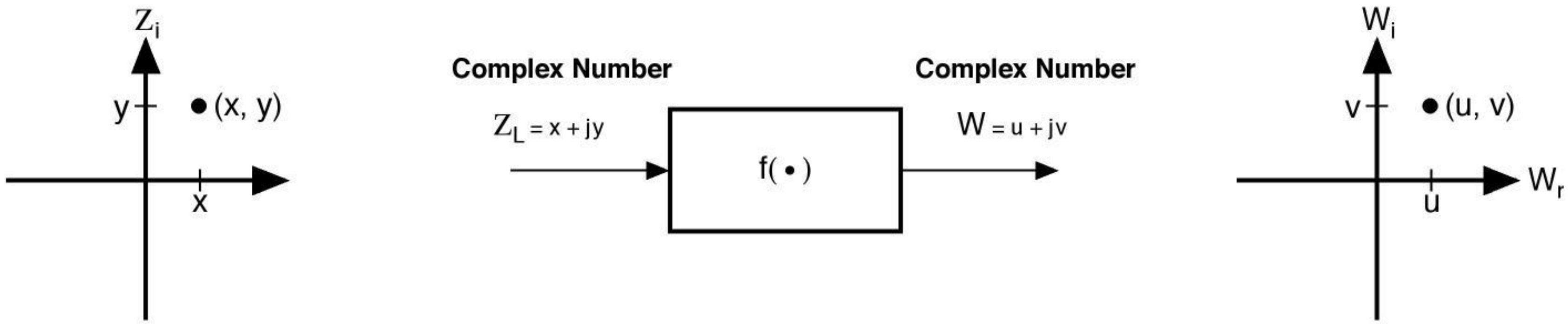
Hewlett Packard 8410, circa 1967

MFJ-259B
circa 1997

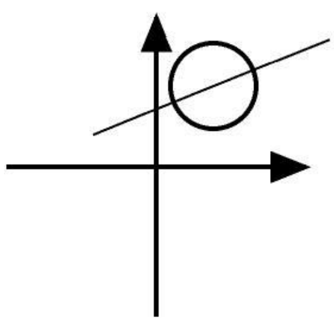
RigExpert AA-2000
circa 2020

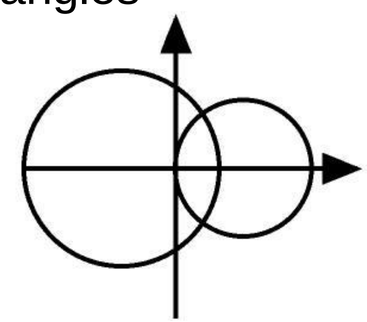
- A little knowledge of basic geometry

Complex Functions



- **Basic types of complex functions**
  - Global Properties
    - Linear – lines map to lines
    - Bilinear – circles map to circles
  - Local Properties
    - Conformal – right angles map to right angles





Bilinear (aka Möbius) Transformations

- **A "rational" function is a ratio of two polynomials**

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_M x^M + \ldots + a_1 x + a_0}{b_N x^N + \ldots + b_1 x + b_0}$$

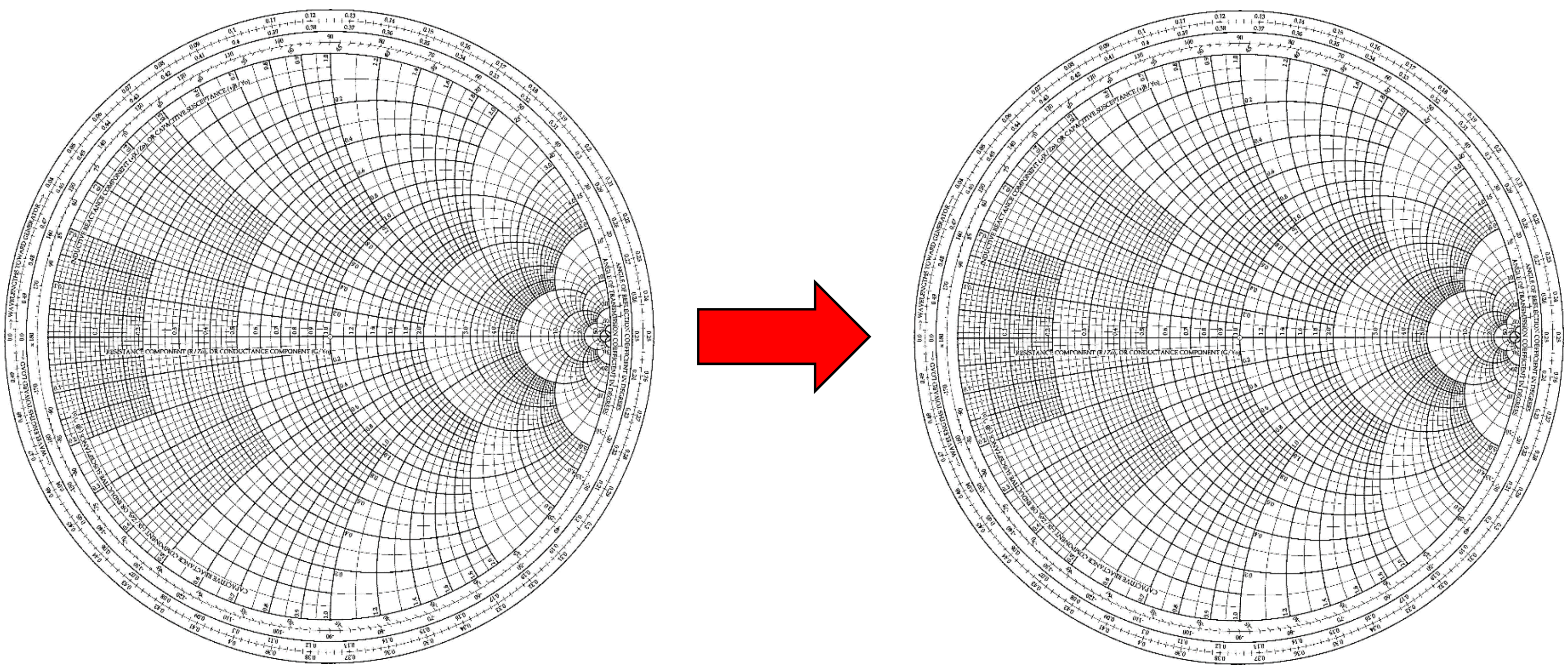
- **A "bilinear" function is a ratio of two 1st degree polynomials**

$$y = \frac{a\,x + b}{c\,x + d}$$

- **When all variables are complex numbers, bilinear functions are**
  - Conformal: angle preserving
  - Map circles to circles
- **The inverse of a bilinear function is also bilinear**

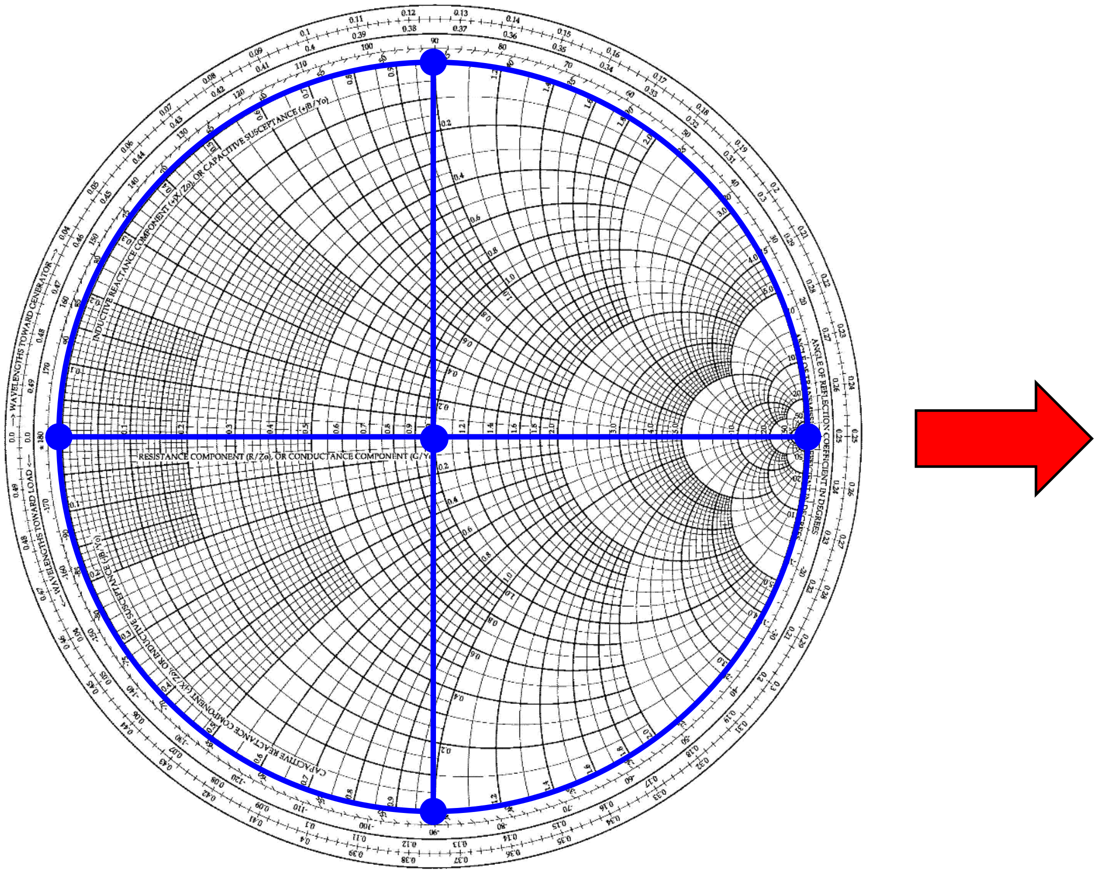
Example: Input Reflection as a function of Load

$$\Gamma_L \longrightarrow \boxed{\frac{S_{11}-\left(S_{11}S_{22}-S_{12}S_{21}\right)\Gamma_L}{1-S_{22}\Gamma_L}} \longrightarrow \Gamma_{IN}$$



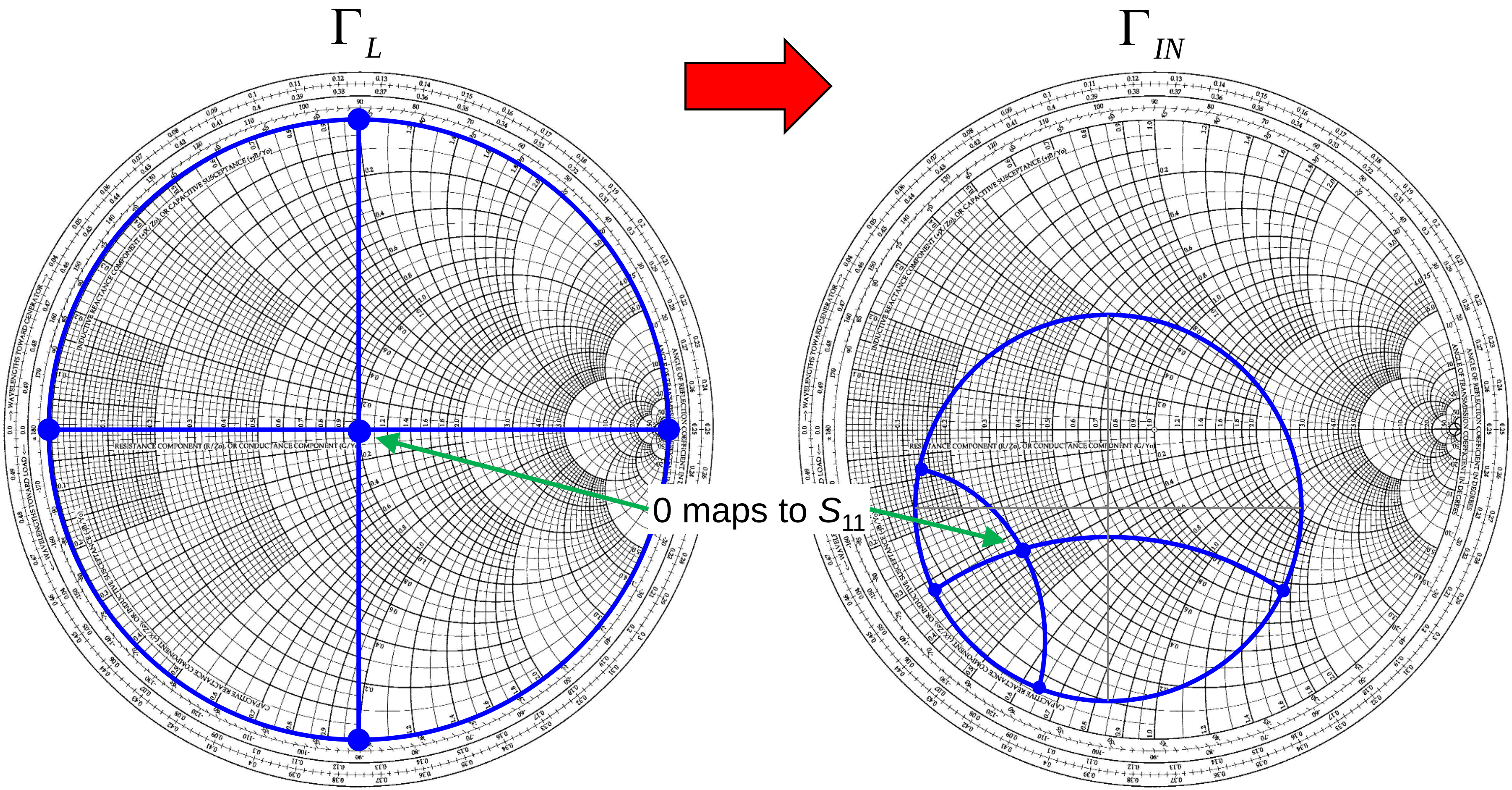
Quiz: Points, Lines, and Circle Map to … ?

Γ_L Γ_{IN}



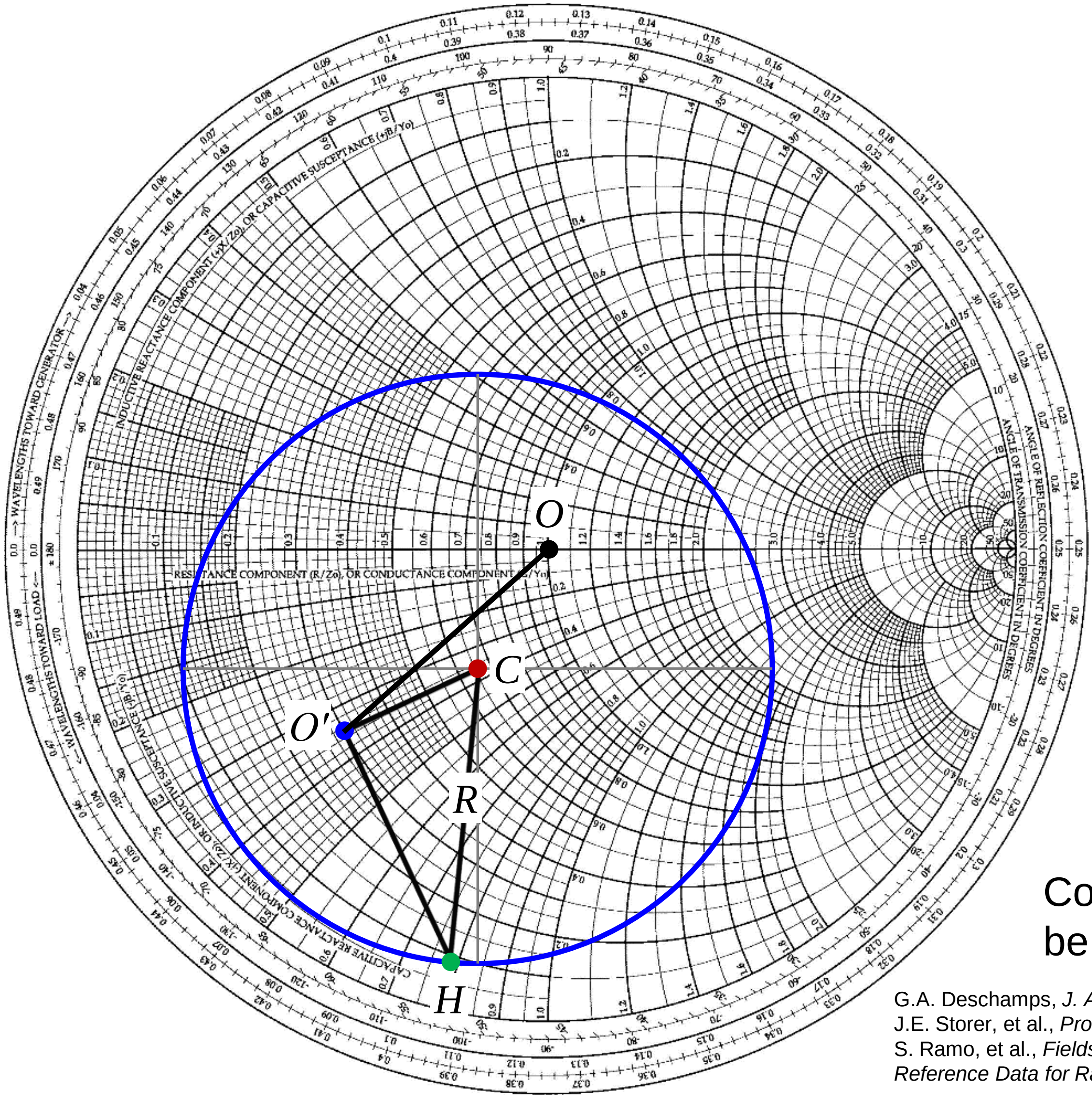
?

Radial Spokes Map to Circular Arcs



Intersections form right angles!

How to Find S Parameter Magnitudes on Smith Chart By Using Straight Edge and Compass



$$R = CH$$

$$S_{11} = O'$$

$$|S_{11}| = OO'$$

$$|S_{22}| = \frac{CO'}{R}$$

$$|S_{21}| = \frac{O'H}{\sqrt{R}}$$

Complex phase angles can likewise be found by geometric constructions

G.A. Deschamps, *J. Appl. Phys.*, Aug. 1953; and *ITT Electrical Communication*, Mar. 1944.
J.E. Storer, et al., *Proc. IRE*, Aug. 1953. Discussion, Sep. 1954.
S. Ramo, et al., *Fields and Waves in Communication Electronics*, Wiley, 1965.
Reference Data for Radio Engineers, 9e, Newnes , 2001. Chapter 31.

Outline of Procedure

- **Measure input reflection coefficient for reactive loads that are diametrically opposite on the load plane Smith chart**
  - You want end points of at least two radials through chart center
  - More than two radials can be used
- **Draw the data points on the input Smith chart**
- **Draw a circle through the data points**
- **Find the center *C* of the circle**
- **For each pair of points, draw the arc that corresponds to its radial**
  - Arcs meet the circle at right angles
- **Find the intersection *O′* of the arcs**
- **Find point *H* on the circle**
- **Compute ratios of distances**

Detailed Geometric Construction

- Put measured data points on Smith chart
- Draw perpendicular bisectors of pairs of points
- Intersection of bisectors gives center *C* of circle
- Draw circle centered on *C* through the data points
- Extend a line from *C* through each data point
- For each data point, draw a perpendicular line through the data point and tangent to the circle
- For the paired data points (Top, Bottom) and (Left, Right), find the intersection point of the two lines tangent to the circle
- Draw circular arcs centered on the intersection points that connect the data points of each pair
- The arcs intersect at the point *O′*
- Draw line segments from *O′* to the centers of the circle *C* and Smith chart *O*
- Draw a line through *O′* perpendicular to *OO′* to find *H*

2-Port Characterization

- **All S parameters of a reciprocal 2-port can be determined from input port reflection measurements alone**
  - Both magnitude and phase angle (modulo 360 degrees)
- **A traditional impedance bridge or single port VNA is sufficient**
- **S_{11} can be found from a single reflection measurement by terminating with a load equal to Port 2 reference impedance (aka a “matched” load)**
- **However, S_{21} and S_{22} generally cannot be found from just two measurements regardless of termination load value**
- **Three reflection measurements, each with different termination load value, are sufficient to determine all S parameters**
- **The method uses measurements with load impedance pairs (at least two pairs)**
  - Loads are reactive, diametrically opposite on Smith chart perimeter, e.g. left and right (0 and ∞) and top and bottom (j 50 and –j 50)
  - Pairs realized by terminating the 2-port with random length lines, themselves alternately terminated by a short circuit or short-circuited eighth-wave stub

Final Comments

- ***S*** **parameters and *ABCD* parameters are useful for analyzing microwave networks involving interconnected 2-port networks, multi-port networks, and cascade chains of 2-port networks**
- **Möbius transformations and Smith charts are important tools for analysis and visualization**
- **Analysis using *S* parameters is done using signal flow graphs and Mason's Rule**
- **VNA instruments directly measure and report *S* parameters for the analyzer's fixed port reference impedance, usually 50 ohms**
- **Input reflection measurements using a traditional impedance bridge or single-port VNA are sufficient to characterize all parameters of a 2-port network**
- **A two-port VNA saves time but is not really necessary**
- ***S*** **parameters can be converted to**
  - *ABCD*, *Z*, *Y*, *H*, or *G* parameters
  - *S* parameters for port reference impedances other than 50 ohms

Further Reading

- David M. Pozar, *Microwave Engineering*, 4th edition, Wiley, 2012.
- George D. Vendelin, Anthony M. Pavio, and Ulrich L. Rohde, *Microwave Circuit Design Using Linear and Nonlinear Techniques*, 2nd edition, Wiley, 2005.
- Robert E. Collin, *Foundations for Microwave Engineering*, 2nd edition, IEEE & Wiley, 2001.
- *Reference Data for Radio Engineers*, 9th edition, Newnes (earlier editions by Sams, ITT, and Federal Telephone and Radio Corp.), 2001.
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- Simon Ramo, John R. Whinnery, and Theodore Van Duzer, *Fields and Waves in Communication Electronics*, 3rd edition, Wiley, 1994.
- Peter A. Rizzi, *Microwave Engineering Passive Circuits*, Prentice Hall, 1988.
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- Möbius transformations: https://en.wikipedia.org/wiki/M%C3%B6bius_transformation
- Stereographic projections: https://en.wikipedia.org/wiki/Stereographic_projection

The End